J Syst Sci Complex (2020) 33: 1733–1754

Time-Inconsistent Stochastic LQ Problem with Regime Switching*

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DOI: 10.1007/s11424-020-9017-z
Received: 14 January 2019 / Revised: 13 June 2019

Abstract This paper investigates a time-inconsistent stochastic linear-quadratic problem with regime switching that is characterized via a finite-state Markov chain. Open-loop equilibrium control is studied in this paper whose existence is characterized via Markov-chain-modulated forward-backward stochastic difference equations and generalized Riccati-like equations with jumps.

Keywords Forward-backward stochastic difference equation, open-loop equilibrium control, regime switching, stochastic linear-quadratic problem, time inconsistency.

1 Introduction

Time inconsistency of this paper is referred to a phenomenon of optimal control: A control which is optimal at some previous time instant is no longer optimal when viewed back in the future. This phenomenon is often observed in dynamic decision makings, and is firstly investigated by Strotz[1] in the 1950s. Strotz's key idea is to view the controller at different instants as different agents, and is to reformulate the time-inconsistent problem as a game between these agents. The equilibrium of this game is a time-consistent solution to the original time-inconsistent optimal control problem.

Since Strotz's work, the game approach is widely accepted, and many practical time-inconsistent scenarios in economics and finance are widely studied; see, for example, [2–5] and references therein. In recent years, there is a growing body of literature from control community

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*This research was supported in part by the National Key R&D Program of China under Grant No. 2018YFA0703800, and by the National Natural Science Foundation of China under Grant Nos. 61773222, 61877057 and 61973172.
◇ *This paper was recommended for publication by Editor LIU Yungang.*

that investigates the time-inconsistent optimal control problems. [6, 7] gave the definition of equilibrium control in continuous time for problems with non-exponential discounting, and [8] discussed the problems of general Markovian time-inconsistent stochastic optimal control. To classify, two kinds of time-consistent equilibrium solutions have been reported within the realm of time-inconsistent optimal control, which are the open-loop equilibrium control and closed-loop equilibrium strategy[9–13]. In [11–15], the researchers investigated Strotz's equilibrium solution[1], and established some theoretical results for time-inconsistent optimal control in terms of closed-loop equilibrium strategy; while [9, 10, 13, 16, 17] studied the open-loop equilibrium control for linear-quadratic (LQ, for short) problems. Recently, [18] proposes a novel notion of equilibrium solution for the time-inconsistent stochastic LQ problem. This notion is called the mixed equilibrium solution, which consists of two parts: a pure-feedback-strategy part and an open-loop-control part. When the pure-feedback-strategy part is zero or the open-loop-control part does not depend on the initial state, the mixed equilibrium solution reduces to the open-loop equilibrium control and feedback equilibrium strategy, respectively. Furthermore, an example is given[18] to show that the mixed equilibrium solution exists for all the initial pairs, although neither the open-loop equilibrium control nor the feedback equilibrium strategy exists for some initial pairs.

LQ problems with regime switching have been extensively studied during the last few decades[19–26]. The regime switching is characterized via a Markov process, which often arises in reality with component failures or repairs, changing subsystem interconnections, and abrupting environmental disturbances; see also [24, 26–30]. Due to their wide existence of time inconsistency and regime switching, it is very necessary to study the LQ problems with both time inconsistency and regime switching. To this aim, in this paper we investigate the open-loop equilibrium control of a time-inconsistent stochastic LQ problem with regime switching, which yet has not been studied before. Necessary and sufficient conditions are derived on the existence of open-loop equilibrium control via a Markov-chain-modulated forward-backward stochastic difference equation (FBSΔE, for short). By decoupling this FBSΔE, conditions in terms of Riccati-like equations with jumps are obtained to characterize the open-loop equilibrium control. In [31], an LQ optimal control is considered for discrete-time Markov jump linear systems and a Markov-chain-modulated forward-backward difference equation (FBΔE, for short) is reported. As [31] deals with a system model without the noise w (of this paper), the FBΔE of [31] differs significantly from the FBSΔE. Therefore, the study of FBSΔE is much involved than that of FBΔE of [31].

The remaining part of this paper is organized as follows. In Section 2, the definition of open-loop equilibrium control is introduced, and its characterization is presented in Section 3. Section 4 gives an illustrative example and Section 5 concludes the paper.

2 Open-Loop Equilibrium Control

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, which is assumed to be abundant enough such that two processes $\theta \triangleq \{\theta_k\}$ and $w \triangleq \{w_k\}$ live on it.

(a) θ is a homogeneous Markov chain taking values in a finite set $\{1, 2, \cdots, \tau\} \triangleq \mathcal{M}$ with a stationary one-step transition probability matrix $\Lambda = (p_{ij}) \in \mathbb{R}^{\tau\times\tau}$. The (i, j)-th entry of Λ is

$$p_{ij} = P(\theta_{k+1} = j|\theta_k = i), \quad i, j \in \mathcal{M}, \quad k = 0, 1, \cdots .$$

The initial distribution of θ_0 is denoted by $\nu = (\nu_1, \nu_2, \cdots, \nu_\tau)^{\mathrm{T}}$ where the superscript T denotes the transposition of a matrix or a vector.

(b) w is a martingale difference sequence in the sense of $\mathbb{E}[w_{k+1}|\mathcal{F}_{k+1}] = 0$ for any k, where $\mathcal{F}_{k+1}$ is the σ-algebra generated by $\{w_\ell, \theta_\ell, \ell = 0, 1, \cdots, k\}$ and $\mathcal{F}_0$ is understood as $\{\emptyset, \Omega\}$. It is also assumed that for any k the process w has the property

$$\mathbb{E}[w_{k+1}^2|\mathcal{F}_{k+1}] = 1,$$

and that θ, w are independent of each other.

Consider the following controlled discrete-time stochastic difference equation (SΔE, for short)

$$\begin{cases} X_{k+1}^t = \big(A_{t,k,\theta_k} X_k^t + B_{t,k,\theta_k} u_k\big) + \big(C_{t,k,\theta_k} X_k^t + D_{t,k,\theta_k} u_k\big) w_k, \\ X_t^t = x, \quad k \in \mathbb{T}_t = \{t, t+1, \cdots, N-1\}, \quad t \in \mathbb{T} = \{0, 1, \cdots, N-1\}, \end{cases} \tag{1}$$

where $\{X_k^t, k \in \widetilde{\mathbb{T}}_t\} \triangleq X^t$ and $\{u_k, k \in \mathbb{T}_t\} \triangleq u$ with $\widetilde{\mathbb{T}}_t = \{t, t+1, \cdots, N\}$ are the state process and control process, respectively; when $\theta_k = i$, the corresponding coefficients $A_{t,k,i}, C_{t,k,i} \in \mathbb{R}^{n\times n}$, $B_{t,k,i}, D_{t,k,i} \in \mathbb{R}^{n\times m}$ are deterministic matrices. In (1), x belongs to $l^2_{\mathcal{F}}(t; \mathbb{R}^n)$ with

$$l^2_{\mathcal{F}}(t; \mathbb{R}^n) = \big\{\zeta \in \mathbb{R}^n \,\big|\, \zeta \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}|\zeta|^2 < \infty\big\}. \tag{2}$$

The cost functional associated with System (1) is

$$J(t, x; u) = \sum_{k=t}^{N-1} \mathbb{E}\big[(X_k^t)^{\mathrm{T}} Q_{t,k,\theta_k} X_k^t + u_k^{\mathrm{T}} R_{t,k,\theta_k} u_k\big] + \mathbb{E}\big[(X_N^t)^{\mathrm{T}} G_{t,\theta_N} X_N^t\big], \tag{3}$$

where for $\theta_k = i$, $Q_{t,k,i}, R_{t,k,i}, k \in \mathbb{T}_t$ and $G_{t,i}$ are deterministic symmetric matrices of appropriate dimensions. Let

$$l^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m) = \big\{\mu = \{\mu_k, k \in \mathbb{T}_t\} \,\big|\, \mu_k \text{ is } \mathcal{F}_k\text{-measurable}, \mathbb{E}|\mu_k|^2 < \infty, k \in \mathbb{T}_t\big\}. \tag{4}$$

Then, we pose the following optimal control problem.

Problem (LQ) For (1), (3) and the initial pair (t, x), find a $u^* \in l^2_{\mathcal{F}}(\mathbb{T}_t; \mathbb{R}^m)$, such that

$$J(t, x; u^*) = \inf_{u \in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)} J(t, x; u).$$

Note that Problem (LQ) is time-inconsistent. The following definition gives a time-consistent solution.

Definition 2.1 $u^{t,x,*} \in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ is called an open-loop equilibrium control of Problem (LQ) for the initial pair (t,x), if

$$J\big(k, X_k^{t,x,*}; u^{t,x,*}|_{\mathbb{T}_k}\big) \le J\big(k, X_k^{t,x,*}; (u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})\big)$$

holds for any $k \in \mathbb{T}_t$ and any $u_k \in l^2_{\mathcal{F}}(k;\mathbb{R}^m)$. Here, $u^{t,x,*}|_{\mathbb{T}_k}$ and $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ are the restrictions of $u^{t,x,*}$ on $\mathbb{T}_k = \{k, k+1, \cdots, N-1\}$ and $\mathbb{T}_{k+1} = \{k+1, k+2, \cdots, N-1\}$, respectively; and $X_k^{t,x,*}$ is computed via

$$\begin{cases} X_{k+1}^{t,x,*} = \big[A_{k,k,\theta_k} X_k^{t,x,*} + B_{k,k,\theta_k} u_k^{t,x,*}\big] + \big[C_{k,k,\theta_k} X_k^{t,x,*} + D_{k,k,\theta_k} u_k^{t,x,*}\big] w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases}$$

3 Solution

Lemma 3.1 *Let $\zeta \in l^2_{\mathcal{F}}(k;\mathbb{R}^n)$, $u = \{u_\ell, k \in \mathbb{T}_k\} \in l^2_{\mathcal{F}}(\mathbb{T}_k;\mathbb{R}^m)$, $\overline{u}_k \in l^2_{\mathcal{F}}(k;\mathbb{R}^m)$ and $\lambda \in \mathbb{R}$. Then, the following equation holds*

$$\begin{aligned} & J(k,\zeta;(u_k + \lambda \overline{u}_k, u|_{\mathbb{T}_{k+1}})) - J(k,\zeta;u) \\ &= 2\lambda \mathbb{E}\Big\{\mathbb{E}_k\big[R_{k,k,\theta_k} u_k + B^{\mathrm{T}}_{k,k,\theta_k} Z^{k,u_k}_{k+1} + D^{\mathrm{T}}_{k,k,\theta_k} Z^{k,u_k}_{k+1} w_k\big]^{\mathrm{T}} \overline{u}_k\Big\} + \lambda^2 \widehat{J}(k,0;\overline{u}_k) \end{aligned}$$

with

$$\widehat{J}(k,0;\overline{u}_k) = \sum_{\ell=k}^{N-1} \mathbb{E}\big[(Y_\ell^{k,\overline{u}_k})^{\mathrm{T}} Q_{k,\ell,\theta_\ell} Y_\ell^{k,\overline{u}_k}\big] + \mathbb{E}\big[\overline{u}_k^{\mathrm{T}} R_{k,\ell,\theta_k} \overline{u}_k\big] + \mathbb{E}\big[(Y_N^{k,\overline{u}_k})^{\mathrm{T}} G_{k,\theta_N} Y_N^{k,\overline{u}_k}\big]. \quad (5)$$

Here, $u|_{\mathbb{T}_{k+1}} = \{u_{k+1}, u_{k+2}, \cdots, u_{N-1}\}$ and the l^2 spaces are similarly defined as those in (2)–(4)*; Z^{k,u_k}, $Y^{k,\overline{u}_k}$ are given, respectively, by the backward stochastic difference equation* (*BSΔE, for short*)

$$\begin{cases} Z_\ell^{k,u_k} = \mathbb{E}_\ell\big[Q_{k,\ell,\theta_\ell} X_\ell^{k,u_k} + A^{\mathrm{T}}_{k,\ell,\theta_\ell} Z^{k,u_k}_{\ell+1} + C^{\mathrm{T}}_{k,\ell,\theta_\ell} Z^{k,u_k}_{\ell+1} w_\ell\big], \\ Z_N^{k,u_k} = \mathbb{E}_N(G_{k,\theta_N} X_N^{k,u_k}), \quad \ell \in \mathbb{T}_t, \end{cases}$$

and the SΔE

$$\begin{cases} Y_{\ell+1}^{k,\overline{u}_k} = A_{k,\ell,\theta_\ell} Y_\ell^{k,\overline{u}_k} + C_{k,\ell,\theta_\ell} Y_\ell^{k,\overline{u}_k} w_\ell, \\ Y_{k+1}^{k,\overline{u}_k} = B_{k,\ell,\theta_k} \overline{u}_k + D_{k,\ell,\theta_k} \overline{u}_k w_k, \\ Y_k^{k,\overline{u}_k} = 0, \quad \ell \in \mathbb{T}_{k+1}, \end{cases}$$

where X_N^k is computed via

$$\begin{cases} X_{\ell+1}^k = \big(A_{k,\ell,\theta_\ell} X_\ell^k + B_{k,\ell,\theta_\ell} u_\ell\big) + \big(C_{k,\ell,\theta_\ell} X_\ell^k + D_{k,\ell,\theta_\ell} u_\ell\big) w_\ell, \\ X_k^k = \zeta, \quad \ell \in \mathbb{T}_k. \end{cases} \quad (6)$$

Proof Replace u_k with $u_k + \lambda\overline{u}_k$ in (6), and denote the solution by $X^{k,\lambda}$. Then,

$$\begin{cases} \dfrac{X^{k,\lambda}_{\ell+1} - X^k_{\ell+1}}{\lambda} = A_{k,\ell,\theta_\ell}\dfrac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda} + C_{k,\ell,\theta_\ell}\dfrac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda}w_\ell, \\ \dfrac{X^{k,\lambda}_{k+1} - X^k_{k+1}}{\lambda} = B_{k,k,\theta_k}\overline{u}_k + D_{k,k,\theta_k}\overline{u}_k w_k, \\ \dfrac{X^{k,\lambda}_k - X^k_k}{\lambda} = 0, \quad \ell \in \mathbb{T}_{k+1}. \end{cases}$$

Denoting $\frac{X^{k,\lambda}_\ell - X^k_\ell}{\lambda}$ by Y^k_ℓ, $\ell \in \mathbb{T}_k$, we get

$$\begin{cases} Y^k_{\ell+1} = A_{k,\ell,\theta_\ell}Y^k_\ell + C_{k,\ell,\theta_\ell}Y^k_\ell w_\ell, \\ Y^k_{k+1} = B_{k,k,\theta_k}\overline{u}_k + D_{k,k,\theta_k}\overline{u}_k w_k, \\ Y^k_k = 0, \quad \ell \in \mathbb{T}_{k+1}. \end{cases}$$

As $X^{k,\lambda}_\ell = X^k_\ell + \lambda Y^k_\ell$, $\ell \in \mathbb{T}_k$, it holds that

$$\begin{aligned} & J(k,\zeta;(u_k + \lambda\overline{u}_k, u|_{\mathbb{T}_{k+1}})) - J(k,\zeta;u) \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}\big[(X^k_\ell + \lambda Y^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell}(X^k_\ell + \lambda Y^k_\ell) - (X^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell} X^k_\ell\big] \\ &\quad + \mathbb{E}\big[(u_k + \lambda\overline{u}_k)^{\mathrm{T}} R_{k,k,\theta_k}(u_k + \lambda\overline{u}_k) - u_k^{\mathrm{T}} R_{k,k,\theta_k} u_k\big] \\ &\quad + \mathbb{E}\big[(X^k_N + \lambda Y^k_N)^{\mathrm{T}} G_{k,\theta_N}(X^k_N + \lambda Y^k_N)\big] - \mathbb{E}\big[(X^k_N)^{\mathrm{T}} G_{k,\theta_N} X^k_N\big] \\ &= 2\lambda\Big\{ \sum_{\ell=k}^{N-1} \mathbb{E}\big[(X^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell} Y^k_\ell\big] + \mathbb{E}\big[u_k^{\mathrm{T}} R_{k,k,\theta_k}\overline{u}_k\big] + \mathbb{E}\big[(X^k_N)^{\mathrm{T}} G_{k,\theta_N} Y^k_N\big] \Big\} \\ &\quad + \lambda^2\Big\{ \sum_{\ell=k}^{N-1} \mathbb{E}\big[(Y^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell} Y^k_\ell\big] + \mathbb{E}\big[\overline{u}_k^{\mathrm{T}} R_{k,k,\theta_k}\overline{u}_k\big] + \mathbb{E}\big[(Y^k_N)^{\mathrm{T}} G_{k,\theta_N} Y^k_N\big] \Big\}. \end{aligned} \tag{7}$$

Note that

$$\begin{aligned} & \sum_{\ell=k}^{N-1} \mathbb{E}\big[(X^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell} Y^k_\ell\big] + \mathbb{E}\big[u_k^{\mathrm{T}} R_{k,k,\theta_k}\overline{u}_k\big] + \mathbb{E}\big[(X^k_N)^{\mathrm{T}} G_{k,\theta_N} Y^k_N\big] \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}\big[(X^k_\ell)^{\mathrm{T}} Q_{k,\ell,\theta_\ell} Y^k_\ell + (Z^k_{\ell+1})^{\mathrm{T}} Y^k_{\ell+1} - (Z^k_\ell)^{\mathrm{T}} Y^k_\ell\big] + \mathbb{E}\big[u_k^{\mathrm{T}} R_{k,k,\theta_k}\overline{u}_k\big] \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}\Big\{ \big[Q_{k,\ell,\theta_\ell} X^k_\ell + A^{\mathrm{T}}_{k,\ell,\theta_\ell} Z^k_{\ell+1} + C^{\mathrm{T}}_{k,\ell,\theta_\ell} Z^k_{\ell+1} w_\ell - Z^k_\ell\big]^{\mathrm{T}} Y^k_\ell \Big\} \\ &\quad + \mathbb{E}\Big[\big(R_{k,k,\theta_k} u_k + B^{\mathrm{T}}_{k,k,\theta_k} Z^{k,u_k}_{k+1} + D^{\mathrm{T}}_{k,k,\theta_k} Z^k_{k+1} w_k\big)^{\mathrm{T}} \overline{u}_k \Big] \\ &= \mathbb{E}\Big\{ \mathbb{E}_k\big[R_{k,k,\theta_k} u_k + B^{\mathrm{T}}_{k,k,\theta_k} Z^{k,u_k}_{k+1} + D^{\mathrm{T}}_{k,k,\theta_k} Z^{k,u_k}_{k+1} w_k\big]^{\mathrm{T}} \overline{u}_k \Big\}. \end{aligned}$$

This together with (7) implies the result. ∎

Concerned with the existence of open-loop equilibrium control, we have the following result whose proof is omitted here due to Lemma 3.1.

Theorem 3.2 *For the initial pair (t,x), the following statements are equivalent.*

(i) *There exists an open-loop equilibrium control of Problem (LQ) for the initial pair (t,x).*

(ii) *The following assertions hold.*

a) *There exists a $u^{t,x,*}\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ such that the stationary condition*

$$\mathbb{E}_k\big[R_{k,k,\theta_k}u_k^{t,x,*}+B^{\mathrm{T}}_{k,k,\theta_k}Z^{k,*}_{k+1}+D^{\mathrm{T}}_{k,k,\theta_k}Z^{k,*}_{k+1}w_k\big]=0,\quad k\in\mathbb{T}_t \tag{8}$$

is satisfied, where $Z^{k,*}_{k+1}$ is computed via the following FBSΔE

$$\begin{cases}X^{k,*}_{\ell+1}=A_{k,\ell,\theta_\ell}X^{k,*}_\ell+B_{k,\ell,\theta_\ell}u^{t,x,*}_\ell+\big(C_{k,\ell,\theta_\ell}X^{k,*}_\ell+D_{k,\ell,\theta_\ell}u^{t,x,*}_\ell\big)w_\ell,\\ Z^{k,*}_\ell=\mathbb{E}_\ell\big[Q_{k,\ell,\theta_\ell}X^{k,*}_\ell+A^{\mathrm{T}}_{k,\ell,\theta_\ell}Z^{k,*}_{\ell+1}+C^{\mathrm{T}}_{k,\ell,\theta_\ell}Z^{k,*}_{\ell+1}w_\ell\big],\\ X^{k,*}_k=X^{t,x,*}_k,\quad Z^{k,*}_N=\mathbb{E}_N(G_{k,\theta_N}X^{k,*}_N),\quad \ell\in\mathbb{T}_k.\end{cases} \tag{9}$$

In (9)*, $X^{t,x,*}_k$ is given by*

$$\begin{cases}X^{t,x,*}_{k+1}=A_{k,k,\theta_k}X^{t,x,*}_k+B_{k,k,\theta_k}u^{t,x,*}_k+\big(C_{k,k,\theta_k}X^{t,x,*}_k+D_{k,k,\theta_k}u^{t,x,*}_k\big)w_k,\\ X^{t,x,*}_t=x,\quad k\in\mathbb{T}_t.\end{cases}$$

b) *The convex condition*

$$\inf_{\overline{u}_k\in l^2_{\mathcal{F}}(k;\mathbb{R}^m)}\widehat{J}(k,0;\overline{u}_k)\geq 0,\quad k\in\mathbb{T}_t \tag{10}$$

is satisfied, where $\widehat{J}(k,0;\overline{u}_k)$ is given in (5).

Under any of the above conditions, $u^{t,x,*}$ given in (ii) *is an open-loop equilibrium control.*

Lemma 3.3 *For $t\in\mathbb{T}_1=\{1,2,\cdots,N-1\}$, it holds that*

$$\mathbb{E}_k[I_{(\theta_k=i)}]=p_{\theta_{k-1}i},\quad k\in\mathbb{T}_t, \tag{11}$$

and

$$\mathbb{E}_k[I_{(\theta_k=i)}w_k]=0,\quad k\in\mathbb{T}_t. \tag{12}$$

Furthermore,

$$\mathbb{E}_0[I_{(\theta_0=i)}]=P(\theta_0=i)=\nu_i,$$

and

$$\mathbb{E}_0[I_{(\theta_0=i)}w_0]=0.$$

Proof Note that the processes θ and w are independent of each other. Concerned with (11) and for $A\in\sigma(w_0,w_1,\cdots,w_{k-1})=\mathcal{F}'_k$, we have

$$\mathbb{E}\big[p_{\theta_{k-1}i}I_A\big]=\mathbb{E}\big[I_A\big]\sum_{j=1}^{\tau}P(\theta_{k-1}=j)p_{ji}=\mathbb{E}\big[I_A\big]P(\theta_k=i)=\mathbb{E}\big[I_{(\theta_k=i)}I_A\big].$$

On the other hand, if $A \in \sigma(\theta_0, \theta_1, \cdots, \theta_{k-1}) = \mathcal{F}''_k$, it holds that

$$\mathbb{E}\big[I_{(\theta_k=i)} I_A\big] = \mathbb{E}\big[I_A \mathbb{E}\big(I_{(\theta_k=i)} | \mathcal{F}''_k\big)\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_A\big].$$

Noting $\mathcal{F}_k = \mathcal{F}'_k \vee \mathcal{F}''_k = \sigma(\mathcal{F}'_k \cup \mathcal{F}''_k)$, we now prove that $\mathbb{E}\big[I_{(\theta_k=i)} I_A\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_A\big]$ holds for any $A \in \mathcal{F}_k$. In fact, letting $A \in \mathcal{F}'_k, B \in \mathcal{F}''_k$, we have

$$\mathbb{E}\big[I_{(\theta_k=i)} I_{A\cap B}\big] = \mathbb{E}\big[I_{(\theta_k=i)} I_A I_B\big] = \mathbb{E}\big[I_{(\theta_k=i)} I_B\big]\mathbb{E}\big[I_A\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_B\big]\mathbb{E}\big[I_A\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_{A\cap B}\big],$$

and

$$\mathbb{E}\big[I_{(\theta_k=i)} I_{A\setminus B}\big] = \mathbb{E}\big[I_{(\theta_k=i)} (I_A - I_{A\cap B})\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_A\big] - \mathbb{E}\big[p_{\theta_{k-1} i} I_{A\cap B}\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_{A\setminus B}\big].$$

Moreover, for $A_i \in \mathcal{F}'_k$ or $A_i \in \mathcal{F}''_k, i = 1, 2, \cdots$, with property $A_i \cap A_j = \emptyset$, $i \neq j$, it holds that

$$\mathbb{E}\big[I_{(\theta_k=i)} I_{\cup_i A_i}\big] = \sum_i \mathbb{E}\big[I_{(\theta_k=i)} I_{A_i}\big] = \sum_i \mathbb{E}\big[p_{\theta_{k-1} i} I_{A_i}\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_{\cup_i A_i}\big].$$

By the above derivation and the definition of σ-algebra, we must have

$$\mathbb{E}\big[I_{(\theta_k=i)} I_A\big] = \mathbb{E}\big[p_{\theta_{k-1} i} I_A\big], \quad A \in \mathcal{F}_k,$$

which implies (11). Furthermore, we can similarly prove (12).

As $\mathcal{F}_0 = \{\emptyset, \Omega\}$, we have

$$\mathbb{E}_0[I_{(\theta_0=i)}] = \mathbb{E}[I_{(\theta_0=i)}] = P(\theta_0 = i) = \nu_i$$

and

$$\mathbb{E}_0[I_{(\theta_0=i)} w_0] = \mathbb{E}[I_{(\theta_0=i)} w_0] = 0.$$

This completes the proof. ∎

Remark 3.4 From Theorem 3.2 and Lemma 3.3, we have that for $t \in \mathbb{T}_1, k \in \mathbb{T}_t$ (8) and (9) become to

$$\begin{aligned} 0 &= \mathbb{E}_k\big[R_{k,k,\theta_k} u_k^{t,x,*} + B_{k,k,\theta_k}^{\mathrm{T}} Z_{k+1}^{k,*} + D_{k,k,\theta_k}^{\mathrm{T}} Z_{k+1}^{k,*} w_k\big] \\ &= \sum_{i=1}^{\tau} p_{\theta_{k-1} i} R_{k,k,i} u_k^{t,x,*} + \sum_{i=1}^{\tau} B_{k,k,i}^{\mathrm{T}} \mathbb{E}_k\big[I_{(\theta_k=i)} Z_{k+1}^{k,*}\big] + \sum_{i=1}^{\tau} D_{k,k,i}^{\mathrm{T}} \mathbb{E}_k\big[I_{(\theta_k=i)} Z_{k+1}^{k,*} w_k\big] \end{aligned} \tag{13}$$

and

$$\begin{cases} X_{\ell+1}^{k,*} = A_{k,\ell,\theta_\ell} X_\ell^{k,*} + B_{k,\ell,\theta_\ell} u_\ell^{t,x,*} + \big(C_{k,\ell,\theta_\ell} X_\ell^{k,*} + D_{k,\ell,\theta_\ell} u_\ell^{t,x,*}\big) w_\ell, \\ Z_\ell^{k,*} = \sum_{i=1}^{\tau} A_{k,\ell,i}^{\mathrm{T}} \mathbb{E}_\ell\big[I_{(\theta_\ell=i)} Z_{\ell+1}^{k,*}\big] + \sum_{i=1}^{\tau} C_{k,\ell,i}^{\mathrm{T}} \mathbb{E}_\ell\big[I_{(\theta_\ell=i)} Z_{\ell+1}^{k,*} w_\ell\big] + \sum_{i=1}^{\tau} p_{\theta_{\ell-1} i} Q_{k,\ell,i} X_\ell^{k,*}, \\ X_k^{k,*} = X_k^{t,x,*}, \quad Z_N^{k,*} = \sum_{i=1}^{\tau} p_{\theta_{N-1} i} G_{k,i} X_N^{k,*}, \quad \ell \in \mathbb{T}_k. \end{cases}$$

If $k = t = 0$, then (13) becomes

$$0 = \sum_{i=1}^{\tau} \nu_i R_{0,0,i} u_0^{0,x,*} + \sum_{i=1}^{\tau} B_{0,0,i}^{\mathrm{T}} \mathbb{E}_0\big[I_{(\theta_0=i)} Z_{0+1}^{0,*}\big] + \sum_{i=1}^{\tau} D_{0,0,i}^{\mathrm{T}} \mathbb{E}_0\big[I_{(\theta_0=i)} Z_{0+1}^{0,*} w_0\big]. \tag{14}$$

Lemma 3.5 *If $u^{t,x,*}$ satisfies (8), then $u_{N-1}^{t,x,*}$ can be selected as*

$$u_{N-1}^{t,x,*} = -W_{N-1,\theta_{N-2}}^{\dagger} H_{N-1,\theta_{N-2}} X_{N-1}^{t,x,*} \tag{15}$$

with property

$$(I - W_{N-1,\theta_{N-2}} W_{N-1,\theta_{N-2}}^{\dagger}) H_{N-1,\theta_{N-2}} X_{N-1}^{t,x,*} = 0,$$

where

$$\begin{cases} W_{N-1,\theta_{N-2}} = \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-2}i} R_{N-1,N-1,i} + \sum_{i=1}^{\tau} p_{\theta_{N-2}i} B_{N-1,N-1,i}^{\mathrm{T}} \Big(\sum_{j=1}^{\tau} p_{ij} G_{N-1,j} \Big) B_{N-1,N-1,i} \\ \qquad\qquad + \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-2}i} D_{N-1,N-1,i}^{\mathrm{T}} \Big(\sum_{j=1}^{\tau} p_{ij} G_{N-1,j} \Big) D_{N-1,N-1,i}, \\ H_{N-1,\theta_{N-2}} = \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-2}i} B_{N-1,N-1,i}^{\mathrm{T}} \Big(\sum_{j=1}^{\tau} p_{ij} G_{N-1,j} \Big) A_{N-1,N-1,i} \\ \qquad\qquad + \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-2}i} D_{N-1,N-1,i}^{\mathrm{T}} \Big(\sum_{j=1}^{\tau} p_{ij} G_{N-1,j} \Big) C_{N-1,N-1,i}. \end{cases}$$

Proof Note that

$$\begin{aligned} Z_N^{N-1,*} &= \sum_{j=1}^{\tau} p_{\theta_{N-1}j} G_{N-1,j} \big(A_{N-1,N-1,\theta_{N-1}} X_{N-1}^{N-1,*} + B_{N-1,N-1,\theta_{N-1}} u_{N-1}^{t,x,*} \big) \\ &\quad + \sum_{j=1}^{\tau} p_{\theta_{N-1}j} G_{N-1,j} \big(C_{N-1,N-1,\theta_{N-1}} X_{N-1}^{N-1,*} + D_{N-1,N-1,\theta_{N-1}} u_{N-1}^{t,x,*} \big) w_{N-1}. \end{aligned}$$

Then,

$$\begin{aligned} &\mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)} Z_N^{N-1,*} \big] \\ &= \sum_{j=1}^{\tau} \mathbb{E}_{N-1}\Big[I_{(\theta_{N-1}=i)} p_{ij} G_{N-1,j} \big(A_{N-1,N-1,i} X_{N-1}^{N-1,*} + B_{N-1,N-1,i} u_{N-1}^{t,x,*} \big) \Big] \\ &\quad + \sum_{j=1}^{\tau} \mathbb{E}_{N-1}\Big[I_{(\theta_{N-1}=i)} p_{ij} G_{N-1,j} \big(C_{N-1,N-1,i} X_{N-1}^{N-1,*} + D_{N-1,N-1,i} u_{N-1}^{t,x,*} \big) w_{N-1} \Big] \\ &= \sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} G_{N-1,j} \big(A_{N-1,N-1,i} X_{N-1}^{N-1,*} + B_{N-1,N-1,i} u_{N-1}^{t,x,*} \big). \end{aligned} \tag{16}$$

In the above, we have used the properties in Lemma 3.3. Furthermore, we have

$$\begin{aligned} &\mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)} Z_N^{N-1,*} w_{N-1} \big] \\ &= \sum_{j=1}^{\tau} \mathbb{E}_{N-1}\Big[I_{(\theta_{N-1}=i)} p_{ij} G_{N-1,j} \big(A_{N-1,N-1,i} X_{N-1}^{N-1,*} + B_{N-1,N-1,i} u_{N-1}^{t,x,*} \big) w_{N-1} \Big] \\ &\quad + \sum_{j=1}^{\tau} \mathbb{E}_{N-1}\Big[I_{(\theta_{N-1}=i)} p_{ij} G_{N-1,j} \big(C_{N-1,N-1,i} X_{N-1}^{N-1,*} + D_{N-1,N-1,i} u_{N-1}^{t,x,*} \big) w_{N-1}^2 \Big] \\ &= \sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} G_{N-1,j} \big(C_{N-1,N-1,i} X_{N-1}^{N-1,*} + D_{N-1,N-1,i} u_{N-1}^{t,x,*} \big). \end{aligned} \tag{17}$$

Hence, it holds that

$$
\begin{aligned}
0 &= \sum_{i=1}^{\tau} p_{\theta_{N-2}i} R_{N-1,N-1,i} u_{N-1}^{t,x,*} + \sum_{i=1}^{\tau} B_{N-1,N-1,i}^{\mathrm{T}} \mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)} Z_N^{N-1,*}\big] \\
&\quad + \sum_{i=1}^{\tau} D_{N-1,N-1,i}^{\mathrm{T}} \mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)} Z_N^{N-1,*} w_{N-1}\big] \\
&= \sum_{i=1}^{\tau} p_{\theta_{N-2}i} R_{N-1,N-1,i} u_{N-1}^{t,x,*} \\
&\quad + \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} B_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} \big(A_{N-1,N-1,i} X_{N-1}^{N-1,*} + B_{N-1,N-1,i} u_{N-1}^{t,x,*}\big) \\
&\quad + \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} D_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} \big(C_{N-1,N-1,i} X_{N-1}^{N-1,*} + D_{N-1,N-1,i} u_{N-1}^{t,x,*}\big) \\
&= \Big\{ \sum_{i=1}^{\tau} p_{\theta_{N-2}i} R_{N-1,N-1,i} + \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} B_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} B_{N-1,N-1,i} \\
&\quad + \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} D_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} D_{N-1,N-1,i} \Big\} u_{N-1}^{t,x,*} \\
&\quad + \Big\{ \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} B_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} A_{N-1,N-1,i} \\
&\quad + \sum_{i=1}^{\tau}\sum_{j=1}^{\tau} p_{\theta_{N-2}i} p_{ij} D_{N-1,N-1,i}^{\mathrm{T}} G_{N-1,j} C_{N-1,N-1,i} \Big\} X_{N-1}^{N-1,*}.
\end{aligned}
$$

From Lemma 3.1 of [32], $u_{N-1}^{t,x,*}$ can be selected as (15). ∎

Lemma 3.6 *Let $t \in \mathbb{T}_1, k \in \mathbb{T}_t$ and assume that $u_\ell^{t,x,*}$ in (8) and (9) has the form $u_\ell^{t,x,*} = \Psi_{\ell,\theta_{\ell-1}} X_\ell^{t,x,*}, \ell \in \mathbb{T}_{k+1}$ with $\Psi_{\ell,\theta_{\ell-1}}$ a matric function of ℓ and $\theta_{\ell-1}$. Then, the backward state $Z^{k,*}$ of (9) is expressed as*

$$
Z_\ell^{k,*} = P_{k,\ell,\theta_{\ell-1}} X_\ell^{k,*} + T_{k,\ell,\theta_{\ell-1}} X_\ell^{t,x,*}, \quad \ell \in \mathbb{T}_{k+1},
$$

where

$$
\begin{cases}
P_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi} Q_{k,\ell,i} + \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} P_{k,\ell+1,i} A_{k,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} P_{k,\ell+1,i} C_{k,\ell,i}, \\
T_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} T_{k,\ell+1,i} A_{\ell,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} T_{k,\ell+1,i} C_{\ell,\ell,i} \\
\qquad\quad + \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} \big(P_{k,\ell+1,i} B_{k,\ell,i} + T_{k,\ell+1,i} B_{\ell,\ell,i}\big) \Psi_{\ell,q} \\
\qquad\quad + \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} \big(P_{k,\ell+1,i} D_{k,\ell,i} + T_{k,\ell+1,i} D_{\ell,\ell,i}\big) \Psi_{\ell,q}, \\
P_{k,N,q} = G_{k,q}, \quad T_{k,N,q} = 0, \\
q = 1, 2, \cdots, \tau, \quad \ell \in \mathbb{T}_{k+1}.
\end{cases}
\tag{18}
$$

Proof Similarly to (16) and (17), we have

$$\mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)}Z_N^{k,*}\big]=\sum_{j=1}^{\tau}p_{\theta_{N-2}i}p_{ij}G_{k,j}\big(A_{k,N-1,i}X_{N-1}^{k,*}+B_{k,N-1,i}u_{N-1}^{t,x,*}\big)$$

and

$$\mathbb{E}_{N-1}\big[I_{(\theta_{N-1}=i)}Z_N^{k,*}w_{N-1}\big]=\sum_{j=1}^{\tau}p_{\theta_{N-2}i}p_{ij}G_{k,j}\big(C_{k,N-1,i}X_{N-1}^{k,*}+D_{k,N-1,i}u_{N-1}^{t,x,*}\big).$$

Hence,

$$\begin{aligned}
Z_{N-1}^{k,*}&=\sum_{i=1}^{\tau}A_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{\theta_{N-2}i}p_{ij}G_{k,j}\big(A_{k,N-1,i}X_{N-1}^{k,*}+B_{k,N-1,i}u_{N-1}^{t,x,*}\big)\\
&\quad+\sum_{i=1}^{\tau}C_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{\theta_{N-2}i}p_{ij}G_{k,j}\big(C_{k,N-1,i}X_{N-1}^{k,*}+D_{k,N-1,i}u_{N-1}^{t,x,*}\big)\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-2}i}Q_{k,N-1,i}X_{N-1}^{k,*}\\
&=\bigg\{\sum_{i=1}^{\tau}p_{\theta_{N-2}i}Q_{k,N-1,i}+\sum_{i=1}^{\tau}p_{\theta_{N-2}i}A_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{ij}G_{k,j}A_{k,N-1,i}\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-2}i}C_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{ij}G_{k,j}C_{k,N-1,i}\big)\bigg\}X_{N-1}^{k,*}\\
&\quad+\bigg\{\sum_{i=1}^{\tau}p_{\theta_{N-2}i}A_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{ij}G_{k,j}B_{k,N-1,i}\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-2}i}C_{k,N-1,i}^{\mathrm{T}}\sum_{j=1}^{\tau}p_{ij}G_{k,j}D_{k,N-1,i}\big)\bigg\}\varPsi_{N-1,\theta_{N-2}}X_{N-1}^{t,x,*}\\
&=P_{k,N-1,\theta_{N-2}}X_{N-1}^{k,*}+T_{k,N-1,\theta_{N-2}}X_{N-1}^{t,x,*}.
\end{aligned}\tag{19}$$

From (19), we obtain

$$\begin{aligned}
&\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)}Z_{N-1}^{k,*}\big]\\
&=\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)}\big(P_{k,N-1,\theta_{N-2}}X_{N-1}^{k,*}+T_{k,N-1,\theta_{N-2}}X_{N-1}^{t,x,*}\big)\big]\\
&=\mathbb{E}_{N-2}\big[P_{k,N-1,i}I_{(\theta_{N-2}=i)}\big(A_{k,N-2,\theta_{N-2}}X_{N-2}^{k,*}+B_{k,N-2,\theta_{N-2}}u_{N-2}^{t,x,*}\big)\big]\\
&\quad+\mathbb{E}_{N-2}\big[T_{k,N-1,i}I_{(\theta_{N-2}=i)}\big(A_{N-2,N-2,\theta_{N-2}}X_{N-2}^{t,x,*}+B_{N-2,N-2,\theta_{N-2}}u_{N-2}^{t,x,*}\big)\big]\\
&=p_{\theta_{N-3}i}P_{k,N-1,i}A_{k,N-2,i}X_{N-2}^{k,*}+p_{\theta_{N-3}i}P_{k,N-1,i}B_{k,N-2,i}\varPsi_{N-2,\theta_{N-3}}X_{N-2}^{t,x,*}\\
&\quad+p_{\theta_{N-3}i}T_{k,N-1,i}\big(A_{N-2,N-2,i}+B_{N-2,N-2,i}\varPsi_{N-2,\theta_{N-3}}\big)X_{N-2}^{t,x,*}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)}Z^{k,*}_{N-1}w_{N-2}\big]\\
&=\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)}\big(P_{k,N-1,\theta_{N-2}}X^{k,*}_{N-1}+T_{k,N-1,\theta_{N-2}}X^{t,x,*}_{N-1}\big)w_{N-2}\big]\\
&=\mathbb{E}_{N-2}\big[P_{k,N-1,i}I_{(\theta_{N-2}=i)}\big(C_{k,N-2,\theta_{N-2}}X^{k,*}_{N-2}+D_{k,N-2,\theta_{N-2}}u^{t,x,*}_{N-2}\big)\big]\\
&\quad+\mathbb{E}_{N-2}\big[T_{k,N-1,i}I_{(\theta_{N-2}=i)}\big(C_{N-2,N-2,\theta_{N-2}}X^{t,x,*}_{N-2}+D_{N-2,N-2,\theta_{N-2}}u^{t,x,*}_{N-2}\big)\big]\\
&=p_{\theta_{N-3}i}P_{k,N-1,i}C_{k,N-2,i}X^{k,*}_{N-2}+p_{\theta_{N-3}i}P_{k,N-1,i}D_{k,N-2,i}\Psi_{N-2,\theta_{N-3}}X^{t,x,*}_{N-2}\\
&\quad+p_{\theta_{N-3}i}T_{k,N-1,i}\big(C_{N-2,N-2,i}+D_{N-2,N-2,i}\Psi_{N-2,\theta_{N-3}}\big)X^{t,x,*}_{N-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z^{k,*}_{N-2}&=\sum_{i=1}^{\tau}A^{\mathrm{T}}_{k,N-2,i}\mathbb{E}_{\ell}\big[I_{(\theta_{N-2}=i)}Z^{k,*}_{N-1}\big]+\sum_{i=1}^{\tau}C^{\mathrm{T}}_{k,N-2,i}\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)}Z^{k,*}_{N-1}w_{N-2}\big]\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}Q_{k,N-2,i}X^{k,*}_{N-2}\\
&=\bigg\{\sum_{i=1}^{\tau}p_{\theta_{N-3}i}Q_{k,N-2,i}+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}A^{\mathrm{T}}_{k,N-2,i}P_{k,N-1,i}A_{k,N-2,i}\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}C^{\mathrm{T}}_{k,N-2,i}P_{k,N-1,i}C_{k,N-2,i}\bigg\}X^{k,*}_{N-2}\\
&\quad+\bigg\{\sum_{i=1}^{\tau}p_{\theta_{N-3}i}A^{\mathrm{T}}_{k,N-2,i}T_{k,N-1,i}A_{N-2,N-2,i}+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}C^{\mathrm{T}}_{k,N-2,i}T_{k,N-1,i}C_{N-2,N-2,i}\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}A^{\mathrm{T}}_{k,N-2,i}\big(P_{k,N-1,i}B_{k,N-2,i}+T_{k,N-1,i}B_{N-2,N-2,i}\big)\Psi_{N-2,\theta_{N-3}}\\
&\quad+\sum_{i=1}^{\tau}p_{\theta_{N-3}i}C^{\mathrm{T}}_{k,N-2,i}\big(P_{k,N-1,i}D_{k,N-2,i}+T_{k,N-1,i}D_{N-2,N-2,i}\big)\Psi_{N-2,\theta_{N-3}}\bigg\}X^{t,x,*}_{N-2}\\
&=P_{k,N-2,\theta_{N-3}}X^{k,*}_{N-2}+T_{k,N-2,\theta_{N-3}}X^{t,x,*}_{N-2}.
\end{aligned}$$

By deduction, we can achieve the conclusion. ∎

Theorem 3.7 *The following statements are equivalent.*

(i) *There exists a* $u^{t,x,*}\in l^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ *such that the stationary condition* (8) *is satisfied.*

(ii) *Either of the following two cases holds.*

a) *For* $t\in\mathbb{T}_1$,

$$(I-W_{k,\theta_{k-1}}W^{\dagger}_{k,\theta_{k-1}})H_{k,\theta_{k-1}}X^{t,x,*}_k=0,\quad k\in\mathbb{T}_t \tag{20}$$

is satisfied, where $X^{t,x,*}$ *is*

$$\begin{cases}X^{t,x,*}_{k+1}=A_{k,k,\theta_k}X^{t,x,*}_k+B_{k,k,\theta_k}u^{t,x,*}_k+\big(C_{k,k,\theta_k}X^{t,x,*}_k+D_{k,k,\theta_k}u^{t,x,*}_k\big)w_k,\\ X^{t,x,*}_t=x,\quad k\in\mathbb{T}_t\end{cases}$$

with

$$u_k^{t,x,*} = -W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} X_k^{t,x,*}, \quad k \in \mathbb{T}_t.$$

In the above, $(W_{k,\theta_{k-1}}, H_{k,\theta_{k-1}})$ *is given by*

$$\begin{cases} W_{k,\theta_{k-1}} = \sum_{i=1}^{\tau} p_{\theta_{k-1}i} \Big[R_{k,k,i} + B_{k,k,i}^{\mathrm{T}} \big(P_{k,k+1,i} + T_{k,k+1,i} \big) B_{k,k,i} \\ \qquad + D_{k,k,i}^{\mathrm{T}} \big(P_{k,k+1,i} + T_{k,k+1,i} \big) D_{k,k,i} \Big], \\ H_{k,\theta_{k-1}} = \sum_{i=1}^{\tau} p_{\theta_{k-1}i} \Big[B_{k,k,i}^{\mathrm{T}} \big(P_{k,k+1,i} + T_{k,k+1,i} \big) A_{k,k,i} \\ \qquad + D_{k,k,i}^{\mathrm{T}} \big(P_{k,k+1,i} + T_{k,k+1,i} \big) C_{k,k,i} \Big], \end{cases} \tag{21}$$

and $(P_{k,k+1,i}, T_{k,k+1,i}), i = 1, 2, \cdots, \tau, k \in \mathbb{T}_t$, *are computed via*

$$\begin{cases} P_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi} Q_{k,\ell,i} + \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} P_{k,\ell+1,i} A_{k,\ell,i} \\ \qquad + \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} P_{k,\ell+1,i} C_{k,\ell,i}, \\ T_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} T_{k,\ell+1,i} A_{\ell,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} T_{k,\ell+1,i} C_{\ell,\ell,i} \\ \qquad - \sum_{i=1}^{\tau} p_{qi} A_{k,\ell,i}^{\mathrm{T}} \big(P_{k,\ell+1,i} B_{k,\ell,i} + T_{k,\ell+1,i} B_{\ell,\ell,i} \big) W_{\ell,q}^{\dagger} H_{\ell,q} \\ \qquad - \sum_{i=1}^{\tau} p_{qi} C_{k,\ell,i}^{\mathrm{T}} \big(P_{k,\ell+1,i} D_{k,\ell,i} + T_{k,\ell+1,i} D_{\ell,\ell,i} \big) W_{\ell,q}^{\dagger} H_{\ell,q}, \\ P_{k,N,q} = G_{k,q}, \quad T_{k,N,q} = 0, \\ q = 1, 2, \cdots, \tau, \quad \ell \in \mathbb{T}_{k+1}. \end{cases} \tag{22}$$

b) *For* $t = 0$,

$$(I - W_{k,\theta_{k-1}} W_{k,\theta_{k-1}}^{\dagger}) H_{k,\theta_{k-1}} X_k^{0,x,*} = 0, \quad k \in \mathbb{T}_1, \tag{23}$$

and

$$(I - W_0 W_0^{\dagger}) H_0 x = 0 \tag{24}$$

are satisfied. Here, $(W_{k,\theta_{k-1}}, H_{k,\theta_{k-1}})$ *is given in* (21) *(with* $k \in \mathbb{T}_1$*), and* (W_0, H_0) *is*

$$\begin{cases} W_0 = \sum_{i=1}^{\tau} \nu_i \big[R_{0,0,i} + B_{0,0,i}^{\mathrm{T}} \big(P_{0,1,i} + T_{0,1,i} \big) B_{0,0,i} + D_{0,0,i}^{\mathrm{T}} \big(P_{0,1,i} + T_{0,1,i} \big) D_{0,0,i} \big], \\ H_0 = \sum_{i=1}^{\tau} \nu_i \big[B_{0,0,i}^{\mathrm{T}} \big(P_{0,1,i} + T_{0,1,i} \big) A_{0,0,i} + D_{0,0,i}^{\mathrm{T}} \big(P_{0,1,i} + T_{0,1,i} \big) C_{0,0,i} \big]. \end{cases}$$

with $(P_{0,1,i}, T_{0,1,i}), i = 1, 2, \cdots, \tau$, *computed via*

$$\begin{cases} P_{0,\ell,q} = \sum_{i=1}^{\tau} p_{qi} Q_{0,\ell,i} + \sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}} P_{0,\ell+1,i} A_{0,\ell,i} \\ \qquad + \sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}} P_{0,\ell+1,i} C_{0,\ell,i}, \\ T_{0,\ell,q} = \sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}} T_{0,\ell+1,i} A_{\ell,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}} T_{0,\ell+1,i} C_{\ell,\ell,i} \\ \qquad - \sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}} \big(P_{0,\ell+1,i} B_{0,\ell,i} + T_{0,\ell+1,i} B_{\ell,\ell,i}\big) W_{\ell,q}^{\dagger} H_{\ell,q} \\ \qquad - \sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}} \big(P_{0,\ell+1,i} D_{0,\ell,i} + T_{0,\ell+1,i} D_{\ell,\ell,i}\big) W_{\ell,q}^{\dagger} H_{\ell,q}, \\ P_{0,N,q} = G_{0,q}, \quad T_{0,N,q} = 0, \\ q = 1, 2, \cdots, \tau, \quad \ell \in \mathbb{T}_1. \end{cases}$$

Further, $X_k^{0,x,*}$ *in* (23) *is computed via*

$$\begin{cases} X_{k+1}^{0,x,*} = A_{k,k,\theta_k} X_k^{0,x,*} + B_{k,k,\theta_k} u_k^{0,x,*} + \big(C_{k,k,\theta_k} X_k^{0,x,*} + D_{k,k,\theta_k} u_k^{0,x,*}\big) w_k, \\ X_0^{0,x,*} = x, \quad k \in \mathbb{T} \end{cases}$$

with

$$u_k^{0,x,*} = \begin{cases} -W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} X_k^{t,x,*}, & k \in \mathbb{T}_1, \\ -W_0^{\dagger} H_0 x, & k = 0. \end{cases}$$

Proof (i)⇒(ii). Firstly consider the case $t \in \mathbb{T}_1$. From Lemma 3.5, letting $\Psi_{N-1,\theta_{N-2}} = -W_{N-1,\theta_{N-2}}^{\dagger} H_{N-1,\theta_{N-2}}$ and substituting it into (18), we have $P_{N-2,N-1,\theta_{N-2}}, T_{N-2,N-1,\theta_{N-2}}$ and

$$Z_{N-1}^{N-2,*} = P_{N-2,N-1,\theta_{N-2}} X_{N-1}^{N-2,*} + T_{N-2,N-1,\theta_{N-2}} X_{N-1}^{t,x,*}.$$

Similarly to (16) and (17), it holds that

$$\begin{aligned} &\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} Z_{N-1}^{N-2,*}\big] \\ &= P_{N-2,N-1,i} \mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} X_{N-1}^{N-2,*}\big] + T_{N-2,N-1,i} \mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} X_{N-1}^{t,x,*}\big] \\ &= p_{\theta_{N-3}i}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) A_{N-2,N-2,i} X_{N-2}^{t,x,*} \\ &\quad + p_{\theta_{N-3}i}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) B_{N-2,N-2,i} u_{N-2}^{t,x,*} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} Z_{N-1}^{N-2,*} w_{N-2}\big] \\ &= p_{\theta_{N-3}i}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) C_{N-2,N-2,i} X_{N-2}^{t,x,*} \\ &\quad + p_{\theta_{N-3}i}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) D_{N-2,N-2,i} u_{N-2}^{t,x,*}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}0 &= \sum_{i=1}^{\tau} p_{\theta_{N-3}i} R_{N-2,N-2,i} u_{N-2}^{t,x,*} + \sum_{i=1}^{\tau} B_{N-2,N-2,i}^{\mathrm{T}} \mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} Z_{N-1}^{N-2,*}\big] \\ &\quad + \sum_{i=1}^{\tau} D_{N-2,N-2,i}^{\mathrm{T}} \mathbb{E}_{N-2}\big[I_{(\theta_{N-2}=i)} Z_{N-1}^{N-2,*} w_{N-2}\big] \\ &= W_{N-2,\theta_{N-3}} u_{N-2}^{t,x,*} + H_{N-2,\theta_{N-3}} X_{N-2}^{t,x,*}, \end{aligned} \tag{25}$$

where

$$\begin{cases} W_{N-2,\theta_{N-3}} = \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-3}i}\big[R_{N-2,N-2,i} + B_{N-2,N-2,i}^{\mathrm{T}}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) B_{N-2,N-2,i} \\ \qquad\qquad + D_{N-2,N-2,i}^{\mathrm{T}}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) D_{N-2,N-2,i}\big], \\ H_{N-2,\theta_{N-3}} = \displaystyle\sum_{i=1}^{\tau} p_{\theta_{N-3}i}\big[B_{N-2,N-2,i}^{\mathrm{T}}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) A_{N-2,N-2,i} \\ \qquad\qquad + D_{N-2,N-2,i}^{\mathrm{T}}\big(P_{N-2,N-1,i} + T_{N-2,N-1,i}\big) C_{N-2,N-2,i}\big]. \end{cases}$$

From Lemma 3.1 of [32] and (25), $u_{N-2}^{t,x,*}$ can be selected as

$$u_{N-2}^{t,x,*} = -W_{N-2,\theta_{N-3}}^{\dagger} H_{N-2,\theta_{N-3}} X_{N-2}^{t,x,*}$$

and

$$(I - W_{N-2,\theta_{N-3}} W_{N-2,\theta_{N-3}}^{\dagger}) H_{N-2,\theta_{N-3}} X_{N-2}^{t,x,*} = 0.$$

By deduction, we can achieve the conclusion.

Consider the case $t = 0$. Similarly to the case $t \in \mathbb{T}_1$, we can prove the results for $k \in \mathbb{T}_1$, and now we pay attention to the result for $k = 0$. Note that

$$Z_{\ell}^{0,*} = P_{0,\ell,\theta_{\ell-1}} X_{\ell}^{0,*} + T_{0,\ell,\theta_{\ell-1}} X_{\ell}^{0,x,*}, \quad \ell \in \mathbb{T}_1,$$

with

$$\begin{cases} P_{0,\ell,q} = \displaystyle\sum_{i=1}^{\tau} p_{qi} Q_{0,\ell,i} + \sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}} P_{0,\ell+1,i} A_{0,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}} P_{0,\ell+1,i} C_{0,\ell,i}, \\ T_{0,\ell,q} = \displaystyle\sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}} T_{0,\ell+1,i} A_{\ell,\ell,i} + \sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}} T_{0,\ell+1,i} C_{\ell,\ell,i} \\ \qquad + \displaystyle\sum_{i=1}^{\tau} p_{qi} A_{0,\ell,i}^{\mathrm{T}}\big(P_{0,\ell+1,i} B_{0,\ell,i} + T_{0,\ell+1,i} B_{\ell,\ell,i}\big) \Psi_{\ell,q} \\ \qquad + \displaystyle\sum_{i=1}^{\tau} p_{qi} C_{0,\ell,i}^{\mathrm{T}}\big(P_{0,\ell+1,i} D_{0,\ell,i} + T_{0,\ell+1,i} D_{\ell,\ell,i}\big) \Psi_{\ell,q}, \\ P_{0,N,q} = G_{0,q}, \quad T_{0,N,q} = 0, \\ q = 1, 2, \cdots, \tau, \quad \ell \in \mathbb{T}_1. \end{cases}$$

Hence,

$$\mathbb{E}_0\big[I_{(\theta_0=i)}Z_1^{0,*}\big]=\nu_i\big(P_{0,1,i}+T_{0,1,i}\big)\big(A_{0,0,i}x+B_{0,0,i}u_0^{0,x,*}\big)$$

and

$$\mathbb{E}_0\big[I_{(\theta_0=i)}Z_1^{0,*}w_0\big]=\nu_i\big(P_{0,1,i}+T_{0,1,i}\big)\big(C_{0,0,i}x+D_{0,0,i}u_0^{0,x,*}\big).$$

Therefore, (14) becomes

$$0=W_0u_0^{0,x,*}+H_0x, \tag{26}$$

where

$$\begin{cases} W_0=\sum_{i=1}^{\tau}\nu_i\big[R_{0,0,i}+B_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i}+T_{0,1,i}\big)B_{0,0,i}+D_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i}+T_{0,1,i}\big)D_{0,0,i}\big],\\ H_0=\sum_{i=1}^{\tau}\nu_i\big[B_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i}+T_{0,1,i}\big)A_{0,0,i}+D_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i}+T_{0,1,i}\big)C_{0,0,i}\big]. \end{cases}$$

From Lemma 3.1 of [32] and (26), $u_0^{t,x,*}$ can be selected as

$$u_0^{0,x,*}=-W_0^{\dagger}H_0x$$

and

$$(I-W_0W_0^{\dagger})H_0x=0.$$

(ii)⇒(i). By reversing the proof of (i)⇒(ii) and Lemma 3.1 of [32], we can achieve the conclusion. ∎

Lemma 3.8 *The following statements are equivalent.*

(i) *The convex condition* (10) *is satisfied.*

(ii) *Either of the following two cases holds.*

a) *For* $t\in\mathbb{T}_1$,

$$\sum_{i=1}^{\tau}p_{\theta_{k-1}i}\big(R_{k,k,i}+B_{k,k,i}^{\mathrm{T}}P_{k,k+1,i}B_{k,k,i}+D_{k,k,i}^{\mathrm{T}}P_{k,k+1,i}D_{k,k,i}\big)\geq 0,\quad k\in\mathbb{T}_t,\ a.s. \tag{27}$$

is satisfied.

b) *For* $t=0$,

$$\sum_{i=1}^{\tau}p_{\theta_{k-1}i}\big(R_{k,k,i}+B_{k,k,i}^{\mathrm{T}}P_{k,k+1,i}B_{k,k,i}+D_{k,k,i}^{\mathrm{T}}P_{k,k+1,i}D_{k,k,i}\big)\geq 0,\quad k\in\mathbb{T}_1,\ a.s. \tag{28}$$

and

$$\sum_{i=1}^{\tau}\nu_i\big(R_{0,0,i}+B_{0,0,i}^{\mathrm{T}}P_{0,1,i}B_{0,0,i}+D_{0,0,i}^{\mathrm{T}}P_{0,1,i}D_{0,0,i}\big)\geq 0 \tag{29}$$

are satisfied.

Furthermore,

$$\sum_{i=1}^{\tau} p_{ji}\big(R_{k,k,i}+B_{k,k,i}^{\rm T}P_{k,k+1,i}B_{k,k,i}+D_{k,k,i}^{\rm T}P_{k,k+1,i}D_{k,k,i}\big)\geq 0,\ j=1,2,\cdots,\tau,\ k\in\mathbb{T}_t \quad (30)$$

implies (27)*. If further the Markov chain θ is irreducible, then* (30) *and* (27) *are equivalent.*

Proof From (5) and the $\{P_{k,\ell,q}\}$ of (22), we have

$$\begin{aligned}
&\widehat{J}(k,0;\overline{u}_k)\\
&=\sum_{\ell=k}^{N-1}\mathbb{E}\Big\{(Y_\ell^{k,\overline{u}_k})^{\rm T}Q_{k,\ell,\theta_\ell}Y_\ell^{k,\overline{u}_k}+(Y_{\ell+1}^{k,\overline{u}_k})^{\rm T}P_{k,\ell+1,\theta_\ell}Y_{\ell+1}^{k,\overline{u}_k}-(Y_\ell^{k,\overline{u}_k})^{\rm T}P_{k,\ell,\theta_{\ell-1}}Y_\ell^{k,\overline{u}_k}\Big\}\\
&\quad+\mathbb{E}\big[\overline{u}_k^{\rm T}R_{k,\ell,\theta_k}\overline{u}_k\big]\\
&=\sum_{\ell=k}^{N-1}\mathbb{E}\Big\{(Y_\ell^{k,\overline{u}_k})^{\rm T}\big[Q_{k,\ell,\theta_\ell}+A_{k,\ell,\theta_\ell}^{\rm T}P_{k,\ell+1,\theta_\ell}A_{k,\ell,\theta_\ell}+C_{k,\ell,\theta_\ell}^{\rm T}P_{k,\ell+1,\theta_\ell}C_{k,\ell,\theta_\ell}-P_{k,\ell,\theta_{\ell-1}}\big]Y_\ell^{k,\overline{u}_k}\Big\}\\
&\quad+\mathbb{E}\big[\overline{u}_k^{\rm T}\big(R_{k,k,\theta_k}+B_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}B_{k,k,\theta_k}+D_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}D_{k,k,\theta_k}\big)\overline{u}_k\big]\\
&\geq 0.
\end{aligned}$$

Hence, the convexity condition (10) is satisfied if and only if

$$\mathbb{E}_k\big(R_{k,k,\theta_k}+B_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}B_{k,k,\theta_k}+D_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}D_{k,k,\theta_k}\big)\geq 0,\ k\in\mathbb{T}_t,\ \text{a.s.}$$

Note that for $t\in\mathbb{T}_1$ and $k\in\mathbb{T}_t$

$$\begin{aligned}
&\mathbb{E}_k\big(R_{k,k,\theta_k}+B_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}B_{k,k,\theta_k}+D_{k,k,\theta_k}^{\rm T}P_{k,k+1,\theta_k}D_{k,k,\theta_k}\big)\\
&=\sum_{i=1}^{\tau}p_{\theta_{k-1}i}\big(R_{k,k,i}+B_{k,k,i}^{\rm T}P_{k,k+1,i}B_{k,k,i}+D_{k,k,i}^{\rm T}P_{k,k+1,i}D_{k,k,i}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_0\big(R_{0,0,\theta_0}+B_{0,0,\theta_0}^{\rm T}P_{0,1,\theta_0}B_{0,0,\theta_0}+D_{0,0,\theta_0}^{\rm T}P_{0,1,\theta_0}D_{0,0,\theta_0}\big)\\
&=\sum_{i=1}^{\tau}\nu_i\big(R_{0,0,i}+B_{0,0,i}^{\rm T}P_{0,1,i}B_{0,0,i}+D_{0,0,i}^{\rm T}P_{0,1,i}D_{0,0,i}\big).
\end{aligned}$$

We then have the equivalence between (i) and (ii).

If the Markov chain θ is irreducible, then $P(\theta_k=j)>0$ for $k\in\mathbb{T}, j=1,2,\cdots,\tau$. Hence, (27) is equivalent to (30). ∎

By the above preparations, we have the following equivalent characterization on the existence of open-loop equilibrium control of Problem (LQ).

Theorem 3.9 *For the initial pair (t,x), the following statements are equivalent.*

(i) *There exists an open-loop equilibrium control of Problem* (*LQ*) *for the initial pair (t,x).*

(ii) *Either of the following two cases holds.*

a) *For $t\in\mathbb{T}_1$, the conditions* (20) *and* (27) *are satisfied.*

b) *For $t=0$, the conditions* (23), (24), (28) *and* (29) *are satisfied.*

So far, we are curious about the case that the initial pair (t, x) rolls out over the product space $\mathbb{T}\times\mathbb{R}^n$. In this case, $k \in \mathbb{T}_t$ in (27) should be slightly changed as $k \in \mathbb{T}$ to characterize the convexity condition. Concerned with the stationary conditions, we can get more beyond (20). For details, see the following theorem.

Theorem 3.10 *The following statements are equivalent.*

(i) *For any initial pair $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, there exists an open-loop equilibrium control of Problem (LQ) for the initial pair (t, x).*

(ii) *The set of difference equations*

$$\begin{cases} \begin{cases} P_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi}Q_{k,\ell,i} + \sum_{i=1}^{\tau} p_{qi}A^{\mathrm{T}}_{k,\ell,i}P_{k,\ell+1,i}A_{k,\ell,i} \\ \qquad + \sum_{i=1}^{\tau} p_{qi}C^{\mathrm{T}}_{k,\ell,i}P_{k,\ell+1,i}C_{k,\ell,i}, \\ T_{k,\ell,q} = \sum_{i=1}^{\tau} p_{qi}A^{\mathrm{T}}_{k,\ell,i}T_{k,\ell+1,i}A_{\ell,\ell,i} + \sum_{i=1}^{\tau} p_{qi}C^{\mathrm{T}}_{k,\ell,i}T_{k,\ell+1,i}C_{\ell,\ell,i} \\ \qquad - \sum_{i=1}^{\tau} p_{qi}A^{\mathrm{T}}_{k,\ell,i}\big(P_{k,\ell+1,i}B_{k,\ell,i} + T_{k,\ell+1,i}B_{\ell,\ell,i}\big)W^{\dagger}_{\ell,q}H_{\ell,q} \\ \qquad - \sum_{i=1}^{\tau} p_{qi}C^{\mathrm{T}}_{k,\ell,i}\big(P_{k,\ell+1,i}D_{k,\ell,i} + T_{k,\ell+1,i}D_{\ell,\ell,i}\big)W^{\dagger}_{\ell,q}H_{\ell,q}, \\ P_{k,N,q} = G_{k,q}, \quad T_{k,N,q} = 0, \\ q = 1, 2, \cdots, \tau, \quad \ell \in \mathbb{T}_{k+1}, \end{cases} \\ \sum_{i=1}^{\tau} p_{\theta_{k-1}i}\big(R_{k,k,i} + B^{\mathrm{T}}_{k,k,i}P_{k,k+1,i}B_{k,k,i} + D^{\mathrm{T}}_{k,k,i}P_{k,k+1,i}D_{k,k,i}\big) \geq 0, \quad k \in \mathbb{T}_1, \\ W_{k,\theta_{k-1}}W^{\dagger}_{k,\theta_{k-1}}H_{k,\theta_{k-1}} = H_{k,\theta_{k-1}}, \quad k \in \mathbb{T}_1, \\ \sum_{i=1}^{\tau} \nu_i\big(R_{0,0,i} + B^{\mathrm{T}}_{0,0,i}P_{0,1,i}B_{0,0,i} + D^{\mathrm{T}}_{0,0,i}P_{0,1,i}D_{0,0,i}\big) \geq 0, \\ W_0W_0^{\dagger}H_0 = H_0 \end{cases} \tag{31}$$

is solvable in the sense of

$$\begin{cases} \sum_{i=1}^{\tau} p_{\theta_{k-1}i}\big(R_{k,k,i} + B^{\mathrm{T}}_{k,k,i}P_{k,k+1,i}B_{k,k,i} + D^{\mathrm{T}}_{k,k,i}P_{k,k+1,i}D_{k,k,i}\big) \geq 0, \quad k \in \mathbb{T}_1, \\ W_{k,\theta_{k-1}}W^{\dagger}_{k,\theta_{k-1}}H_{k,\theta_{k-1}} = H_{k,\theta_{k-1}}, \quad k \in \mathbb{T}_1, \\ \sum_{i=1}^{\tau} \nu_i\big(R_{0,0,i} + B^{\mathrm{T}}_{0,0,i}P_{0,1,i}B_{0,0,i} + D^{\mathrm{T}}_{0,0,i}P_{0,1,i}D_{0,0,i}\big) \geq 0, \\ W_0W_0^{\dagger}H_0 = H_0, \end{cases}$$

where

$$\begin{cases} W_{k,\theta_{k-1}} = \sum_{i=1}^{\tau} p_{\theta_{k-1}i}\Big[R_{k,k,i} + B_{k,k,i}^{\mathrm{T}}\big(P_{k,k+1,i} + T_{k,k+1,i}\big)B_{k,k,i} \\ \qquad\qquad + D_{k,k,i}^{\mathrm{T}}\big(P_{k,k+1,i} + T_{k,k+1,i}\big)D_{k,k,i}\Big], \\ H_{k,\theta_{k-1}} = \sum_{i=1}^{\tau} p_{\theta_{k-1}i}\Big[B_{k,k,i}^{\mathrm{T}}\big(P_{k,k+1,i} + T_{k,k+1,i}\big)A_{k,k,i} \\ \qquad\qquad + D_{k,k,i}^{\mathrm{T}}\big(P_{k,k+1,i} + T_{k,k+1,i}\big)C_{k,k,i}\Big], \\ k \in \mathbb{T}_1, \end{cases}$$

and

$$\begin{cases} W_0 = \sum_{i=1}^{\tau} \nu_i\big[R_{0,0,i} + B_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i} + T_{0,1,i}\big)B_{0,0,i} + D_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i} + T_{0,1,i}\big)D_{0,0,i}\big], \\ H_0 = \sum_{i=1}^{\tau} \nu_i\big[B_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i} + T_{0,1,i}\big)A_{0,0,i} + D_{0,0,i}^{\mathrm{T}}\big(P_{0,1,i} + T_{0,1,i}\big)C_{0,0,i}\big]. \end{cases}$$

Under any of above conditions,

$$u_k^{t,x,*} = \begin{cases} \begin{cases} -W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} X_k^{t,x,*}, & k \in \mathbb{T}_1, \\ -W_0^{\dagger} H_0 x, & k = 0, \end{cases} & t = 0, \\ -W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} X_k^{t,x,*}, & t \in \mathbb{T}_1,\ k \in \mathbb{T}_t \end{cases}$$

is an open-loop equilibrium control of Problem (LQ) *for the initial pair* (t,x)*, and the corresponding open-loop equilibrium state is*

$$\begin{cases} X_{k+1}^{t,x,*} = A_{k,k,\theta_k} X_k^{t,x,*} + B_{k,k,\theta_k} u_k^{t,x,*} + \big(C_{k,k,\theta_k} X_k^{t,x,*} + D_{k,k,\theta_k} u_k^{t,x,*}\big)w_k, \\ X_t^{t,x,*} = x, \quad k \in \mathbb{T}_t. \end{cases}$$

Proof ii)⇒i). This follows from Theorem 3.7 and Lemma 3.8.

i)⇒ii). Note (20). Letting $k = t$ and taking different $x's$, we have $W_{k,\theta_{k-1}} W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} = H_{k,\theta_{k-1}}, k \in \mathbb{T}$. As for any (t,x) with $t \in \mathbb{T}$ and $x \in l_{\mathcal{F}}^2(t;\mathbb{R}^n)$ Problem $(\mathrm{LQ})_{tx}$ admits an open-loop equilibrium control, we must have the solvability of (31). ∎

Remark 3.11 If the Markov chain θ is irreducible, then $P(\theta_k = j) > 0$ for $k \in \mathbb{T}, j = 1,2,\cdots,\tau$. In this case, if

$$\begin{cases} \sum_{i=1}^{\tau} p_{ji}\big(R_{k,k,i} + B_{k,k,i}^{\mathrm{T}} P_{k,k+1,i} B_{k,k,i} + D_{k,k,i}^{\mathrm{T}} P_{k,k+1,i} D_{k,k,i}\big) \geq 0, \\ W_{k,j} W_{k,j}^{\dagger} H_{k,j} = H_{k,j}, \quad k \in \mathbb{T}_1, \\ j = 1,2,\cdots,\tau \end{cases}$$

is satisfied, then

$$\begin{cases} \sum_{i=1}^{\tau} p_{\theta_{k-1}i}\big(R_{k,k,i} + B_{k,k,i}^{\mathrm{T}} P_{k,k+1,i} B_{k,k,i} + D_{k,k,i}^{\mathrm{T}} P_{k,k+1,i} D_{k,k,i}\big) \geq 0, \\ W_{k,\theta_{k-1}} W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} = H_{k,\theta_{k-1}}, \quad k \in \mathbb{T}_1 \end{cases}$$

will hold.

4 An Example

Consider an example of Problem (LQ) with the system equation

$$\begin{cases} X_{k+1} = \big(A_{k,\theta_k} X_k + B_{k,\theta_k} u_k\big) + \big(C_{k,\theta_k} X_k^t + D_{k,\theta_k} u_k\big) w_k, \\ X_t = x, \quad k \in \mathbb{T}_t = \{t, \cdots, 2\}, \quad t \in \mathbb{T} = \{0, 1, 2\}, \end{cases}$$

and the cost functional

$$J(t, x; u) = \sum_{k=t}^{2} \mathbb{E}\big[(X_k)^{\mathrm{T}} Q_{t,k,\theta_k} X_k + u_k^{\mathrm{T}} R_{t,k,\theta_k} u_k\big] + \mathbb{E}\big[X_3^{\mathrm{T}} G_{t,\theta_3} X_3\big],$$

where

$$\begin{aligned}
&A_{0,\theta_0|_{\theta_0=1}} = 1.1, \quad A_{0,\theta_0|_{\theta_0=2}} = 0.51, \quad A_{1,\theta_1|_{\theta_1=1}} = -1.41, \quad A_{1,\theta_1|_{\theta_1=2}} = -1.5, \\
&A_{2,\theta_2|_{\theta_2=1}} = 2.1, \quad A_{2,\theta_2|_{\theta_2=2}} = -1.55, \quad B_{0,\theta_0|_{\theta_0=1}} = -1.5, \quad B_{0,\theta_0|_{\theta_0=2}} = 1.35, \\
&B_{1,\theta_1|_{\theta_1=1}} = -1.5, \quad B_{1,\theta_1|_{\theta_1=2}} = -1.85, \quad B_{2,\theta_2|_{\theta_2=1}} = 0, \quad B_{2,\theta_2|_{\theta_2=2}} = 2.55, \\
&C_{0,\theta_0|_{\theta_0=1}} = 2.14, \quad C_{0,\theta_0|_{\theta_0=2}} = -1.31, \quad C_{1,\theta_1|_{\theta_1=1}} = -2.431, \quad C_{1,\theta_1|_{\theta_1=2}} = 2.38, \\
&C_{2,\theta_2|_{\theta_2=1}} = -2.341, \quad C_{2,\theta_2|_{\theta_2=2}} = 2.445, \quad D_{0,\theta_0|_{\theta_0=1}} = 1.455, \quad D_{0,\theta_0|_{\theta_0=2}} = -2.345, \\
&D_{1,\theta_1|_{\theta_1=1}} = 2.533, \quad D_{1,\theta_1|_{\theta_1=2}} = 2.45, \quad D_{2,\theta_2|_{\theta_2=1}} = 1.5, \quad D_{2,\theta_2|_{\theta_2=2}} = 0, \\
&Q_{0,0,\theta_0|_{\theta_0=1}} = Q_{0,0,\theta_0|_{\theta_0=2}} = 2, \quad Q_{0,1,\theta_1|_{\theta_1=1}} = Q_{0,1,\theta_1|_{\theta_1=2}} = 0, \\
&Q_{0,2,\theta_2|_{\theta_2=1}} = Q_{0,2,\theta_2|_{\theta_2=2}} = 1, \quad Q_{1,1,\theta_1|_{\theta_1=1}} = Q_{1,1,\theta_1|_{\theta_1=2}} = 1.5, \\
&Q_{1,2,\theta_2|_{\theta_2=1}} = Q_{1,2,\theta_2|_{\theta_2=2}} = 1.75, \quad Q_{2,2,\theta_2|_{\theta_2=1}} = Q_{2,2,\theta_2|_{\theta_2=2}} = 1, \\
&R_{0,0,\theta_0|_{\theta_0=1}} = R_{0,0,\theta_0|_{\theta_0=2}} = 1, \quad R_{0,1,\theta_1|_{\theta_1=1}} = R_{0,1,\theta_1|_{\theta_1=2}} = 2.5, \\
&R_{0,2,\theta_2|_{\theta_2=1}} = R_{0,2,\theta_2|_{\theta_2=2}} = 2, \quad R_{1,1,\theta_1|_{\theta_1=1}} = R_{1,1,\theta_1|_{\theta_1=2}} = 2, \\
&R_{1,2,\theta_2|_{\theta_2=1}} = R_{1,2,\theta_2|_{\theta_2=2}} = 3, \quad R_{2,2,\theta_2|_{\theta_2=1}} = R_{2,2,\theta_2|_{\theta_2=2}} = 3.45, \\
&G_{0,\theta_3|_{\theta_3=1}} = 1, \quad G_{0,\theta_3|_{\theta_3=2}} = 1.5, \quad G_{1,\theta_3|_{\theta_3=1}} = 2, \quad G_{1,\theta_3|_{\theta_3=2}} = 2.5, \\
&G_{2,\theta_3|_{\theta_3=1}} = 0.5, \quad G_{2,\theta_3|_{\theta_3=2}} = 1.75.
\end{aligned}$$

Here, Markov chain θ takes values in $\mathcal{M} = \{1, 2\}$ with the transition probability matrix

$$\Lambda = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and the initial distribution of θ is $\nu = (\frac{1}{2}, \frac{1}{2})$.

Solution By some calculations, we have

$$\begin{aligned}
&\sum_{i=1}^{2} p_{1i}\big(R_{2,2,i} + B_{2,i}^{\mathrm{T}} P_{2,3,i} B_{2,i} + D_{2,i}^{\mathrm{T}} P_{2,3,i} D_{2,i}\big) = 12.2658 > 0,\\
&\sum_{i=1}^{2} p_{2i}\big(R_{2,2,i} + B_{2,i}^{\mathrm{T}} P_{2,3,i} B_{2,i} + D_{2,i}^{\mathrm{T}} P_{2,3,i} D_{2,i}\big) = 9.7022 > 0,\\
&\sum_{i=1}^{2} p_{1i}\big(R_{1,1,i} + B_{1,i}^{\mathrm{T}} P_{1,2,i} B_{1,i} + D_{1,i}^{\mathrm{T}} P_{1,2,i} D_{1,i}\big) = 206.8808 > 0,\\
&\sum_{i=1}^{2} p_{2i}\big(R_{1,1,i} + B_{1,i}^{\mathrm{T}} P_{1,2,i} B_{1,i} + D_{1,i}^{\mathrm{T}} P_{1,2,i} D_{1,i}\big) = 203.3189 > 0,\\
&\sum_{i=1}^{2} \nu_i\big(R_{0,0,i} + B_{0,i}^{\mathrm{T}} P_{0,1,i} B_{0,i} + D_{0,i}^{\mathrm{T}} P_{0,1,i} D_{0,i}\big) = 578.8061 > 0,\\
&W_{2,1} = 12.2658 \neq 0, \quad W_{2,2} = 9.7022 \neq 0, \quad W_{1,1} = 170.9644 \neq 0,\\
&W_{1,2} = 167.4800 \neq 0, \quad W_0 = 467.8447 \neq 0.
\end{aligned}$$

According to this and Remark 3.11, we have that the corresponding (31) is solvable. Therefore, for any $(t, x) \in \{0, 1, 2\} \times \mathbb{R}$, the considered LQ problem admits an open-loop equilibrium control. For $(0, x)$, the control

$$u^{0,x,*} = \begin{cases} -W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}} X_k^{0,x,*}, & k \in \{1, 2\},\\ -W_0^{\dagger} H_0 x, & k = 0, \end{cases}$$

is an open-loop equilibrium control, where

$$\begin{aligned}
&-W_0^{\dagger} H_0 = -0.4537, \quad -W_{1,1}^{\dagger} H_{1,1} = -0.5826, \quad -W_{1,2}^{\dagger} H_{1,2} = -0.2493,\\
&-W_{2,1}^{\dagger} H_{2,1} = 0.4587, \quad -W_{2,2}^{\dagger} H_{2,2} = 0.4469,
\end{aligned}$$

and $X^{0,x,*}$ is computed via

$$\begin{cases} X_{k+1}^{0,x,*} = \big(A_{k,\theta_k} - B_{k,\theta_k} W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}}\big) X_k^{0,x,*}\\ \qquad\qquad + \big(C_{k,\theta_k} D_{k,\theta_k} W_{k,\theta_{k-1}}^{\dagger} H_{k,\theta_{k-1}}\big) X_k^{0,x,*} w_k, \quad k = 1, 2,\\ X_1^{0,x,*} = \big(A_{0,\theta_0} - B_{0,\theta_0} W_0^{\dagger} H_0\big) X_0^{0,x,*}\\ \qquad\qquad + \big(C_{0,\theta_0} - D_{0,\theta_0} W_0^{\dagger} H_0\big) X_0^{0,x,*} w_0,\\ X_0^{0,x,*} = x. \end{cases}$$

5 Conclusion

In this paper, we investigated the open-loop equilibrium control for a time-inconsistent stochastic LQ problem with regime switching. Necessary and sufficient conditions are presented to characterize the existence of open-loop equilibrium control via the Markov-chain-modulated

FBSΔE and generalized Riccati-like equations. For future researches, we would like to extend the methodology developed in this paper to other types of time inconsistency.

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